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HIGHER ORDER MIXING AND RIGIDITY OF ALGEBRAIC ACTIONS ON COMPACT ABELIAN GROUPS

BY

SIDDHARTHA BHATTACHARYA

School of Mathematics, Tata Institute of Fundamental Research
Homi Bhabha Road, Mumbai 400005, India
e-mail: siddhart@math.tifr.res.in

ABSTRACT

Let Γ be a discrete group and for $i = 1, 2$; let α_i be an action of Γ on a compact abelian group X_i by continuous automorphisms of X_i. We study measurable equivariant maps $f\colon (X_1, \alpha_1) \to (X_2, \alpha_2)$, and prove a rigidity result under certain assumption on the order of mixing of the underlying actions.

1. Introduction

Throughout this paper the term **compact abelian group** will denote an infinite compact metrizable abelian group. An **algebraic action** α of a discrete group Γ on a compact abelian group X is a homomorphism $\alpha\colon \gamma \mapsto \alpha(\gamma)$, from Γ into the group $\mathrm{Aut}(X)$ of continuous automorphisms of X. An algebraic Γ-action α on a compact abelian group X is said to be **mixing of order** k if for any k Borel sets $A_1, \ldots, A_k \subset X$, and for any k sequences $s_1, \ldots, s_k$ in Γ with $s_i^{-1} s_j(n) \to \infty$ as $n \to \infty$ for all $i \neq j$,

$$\lim_{n\to\infty} \lambda_X\left(\bigcap_{i=1}^{k} \alpha(s_i(n))(A_i)\right) = \prod_{i=1}^{k} \lambda_X(A_i).$$

The action α is said to be **mixing** if it is mixing of order two.

Let α and β be algebraic Γ-actions on compact abelian groups X and Y, respectively. A Borel map $\phi\colon X \longrightarrow Y$ is **equivariant** if

$$\phi \circ \alpha(\gamma) = \beta(\gamma) \circ \phi \ \lambda_X\text{-a.e.}, \quad \text{for every } \gamma \in \Gamma.$$

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The actions α and β are **measurably, topologically** or **algebraically conjugate** if the map ϕ as above can be chosen to be a Borel isomorphism, a homeomorphism or a continuous group isomorphism (in which case ϕ is called a **measurable, topological** or **algebraic conjugacy** of (X,α) and (Y,β)). A map ψ: $X \longrightarrow Y$ is **affine** if there exist a continuous group homomorphism ψ': $X \longrightarrow Y$ and an element $y \in Y$ with $\psi(x) = \psi'(x) + y$ for every $x \in X$.

In this paper we study rigidity properties of equivariant measurable maps between two algebraic Γ-actions. In order to formulate our main result, we need to introduce the notion of a measurable polynomial between compact abelian groups. Let X_1 and X_2 be compact abelian groups and let f be a measurable map from X_1 to X_2. For any h in X_1, we define a map $D_h(f)$: $X_1 \to X_2$ by

$$D_h(f)(x) = f(hx)f(x)^{-1}.$$

The map f is said to be a **measurable polynomial** if there exists a positive integer k such that for all $h_1, \dots, h_{k+1}$ in X_1,

$$D_{h_1} \circ \cdots \circ D_{h_{k+1}}(f)(x) = 1 \quad \text{for } \lambda_{X_1}\text{-a.e. } x \in X_1.$$

The smallest such integer is said to be the **degree** of f and it will be denoted by $\deg(f)$. Our notion of a measurable polynomial can be viewed as the measurable analogue of the the notion of a group polynomial, as defined in [4]. We will show that any measurable polynomial agrees almost everywhere with a continuous map, and that any measurable polynomial between two connected compact abelian groups agrees almost everywhere with an affine map (see Proposition 3.3).

We will prove the following result.

THEOREM 1.1: *Let Γ be a discrete group and let (X_1, α_1) and (X_2, α_2) be algebraic Γ-actions on compact abelian groups X_1 and X_2, respectively. Suppose furthermore that the action α_1 is mixing and there exists an integer $k \geq 2$ with the following property: for every non-trivial closed α_2-invariant subgroup $H \subset X_2$, the restriction of α_2 to H is not $k+1$-mixing. Then any measurable Γ-equivariant map f: $X_1 \to X_2$ is a measurable polynomial with* $\deg(f) \leq k-1$.

In the special case when $\Gamma = \mathbb{Z}^d$ and X_1, X_2 are zero-dimensional, we will prove the following result as a consequence of the above theorem.

COROLLARY 1.2: *Let α_1 and α_2 be two mixing zero-entropy algebraic $\mathbb{Z}^d$-actions on compact zero-dimensional abelian groups X_1 and X_2, respectively. Then for any measurable equivariant map f: $X_1 \to X_2$ there exists a continuous map*

$h: X_1 \to X_2$ such that $f = h$ a.e. λ_{X_1}. In particular, if α_1 and α_2 are measurably conjugate then they are topologically conjugate.

We remark that under additional assumptions on the actions α_1 and α_2, it can be shown that the map f agrees a.e. with an affine map. For various related results and examples see [1], [2], [3] and [5].

It is known that any mixing algebraic $\mathbb{Z}^d$-action on a connected compact abelian group is mixing of all orders (cf. [7]). However, if Γ is a torsion-free discrete group which is not virtually abelian, then we will show that for any mixing algebraic Γ-action on a compact connected finite-dimensional abelian group X, there exists $k \geq 2$ with the property stated in Theorem 1.1 (see Proposition 2.5). Since any measurable polynomial between connected groups agrees almost everywhere with an affine map, as another consequence of Theorem 1.1 we obtain the following result.

COROLLARY 1.3: *Let Γ be a torsion-free discrete group which is not virtually abelian, and let α_1 and α_2 be mixing algebraic Γ-actions on compact connected finite-dimensional abelian groups X_1 and X_2, respectively. Then any measurable Γ-equivariant map $f: X_1 \to X_2$ agrees almost everywhere with an affine map. In particular, if the actions α_1 and α_2 are measurably conjugate then they are algebraically conjugate.*

2. Mixing properties of algebraic actions

For any compact abelian group X, we denote by $\widehat{X}$ the Pontryagin dual of X. If Γ is a discrete group and (X, α) is an algebraic Γ-action, then we denote by $\widehat{\alpha}$ the Γ-action on $\widehat{X}$ induced by α.

In this section we collect some results on higher order mixing of algebraic Γ-actions. We begin with the following Lemma which shows that the order of mixing of an algebraic Γ-action (X, α) can be described in terms of the dual action $\widehat{\alpha}$. The proof involves the standard technique of approximating indicator functions by linear combinations of characters, and is left to the reader.

LEMMA 2.1: *Let Γ be a discrete group and let (X, α) be an algebraic Γ-action. Then for any $k \geq 2$, the following are equivalent.*

1. *The action α is not mixing of order $k+1$.*
2. *There exist $\phi_0, \ldots, \phi_k$ in $\widehat{X}$, and sequences $s_1, \ldots, s_k$ in Γ such that $\phi_0 \neq 0$, $s_i(j) \to \infty$ as $j \to \infty$ for each $i = 1, \ldots, k$, and*

$$\phi_0 = \sum_{i=1}^{k} \widehat{\alpha}(s_i(j))(\phi_i)$$

for every $j \geq 1$.

We remark that if α is k-mixing then we can choose non-zero $\phi_0, \dots, \phi_k$ in $\widehat{X}$, and sequences $s_1, \dots, s_k$ in Γ with $s_i^{-1}s_j(n) \to \infty$ as $n \to \infty$ for all $i \neq j$, which satisfy the second condition in the above lemma.

For $d \geq 1$, by $R_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ we denote the ring of Laurent polynomials with integral coefficients in the variables $u_1, \dots, u_d$ and write the elements $f \in R_d$ as

$$f = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} u^{\mathbf{n}} \tag{2.1}$$

with $u^{\mathbf{n}} = u_1^{n_1} \cdots u_d^{n_d}$ and $f_{\mathbf{n}} \in \mathbb{Z}$ for all $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$, where $f_{\mathbf{n}} = 0$ for all but finitely many $\mathbf{n} \in \mathbb{Z}^d$.

If α is an algebraic $\mathbb{Z}^d$-action on a compact abelian group X, then the additively-written dual group $M = \widehat{X}$ is a module over the ring R_d with respect to the operation

$$f \cdot a = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} \widehat{\alpha}(\mathbf{n})(a)$$

for $f \in R_d$ and $a \in M$. The module $M = \widehat{X}$ is called the **dual module** of α. Conversely, if M is a module over R_d, then we obtain an algebraic $\mathbb{Z}^d$-action α_M on $X_M = \widehat{M}$ by setting

$$\widehat{\alpha_M}(\mathbf{n})(a) = u^{\mathbf{n}} \cdot a$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $a \in M$. Clearly, M is the dual module of α_M.

A prime ideal $\mathfrak{p} \subset R_d$ is **associated with** M if there exists $m \in M$ with $\mathfrak{p} = \{f \in R_d | \, f \cdot m = 0\}$. By $\mathrm{Asc}(M)$ we denote the set of prime ideals associated to M. If M is Noetherian then $\mathrm{Asc}(M)$ is finite. It is known that various dynamical properties of α_M can largely be determined from $\mathrm{Asc}(M)$. We recall the following results (cf. [6, Propositions 6.6 and 6.9]).

LEMMA 2.2: *Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X. Then the group X is zero-dimensional if and only if every prime ideal $\mathfrak{p}$ associated with the dual module $M = \widehat{X}$ of α contains a rational prime constant $p(\mathfrak{p}) > 1$.*

LEMMA 2.3: *Let α be an algebraic $\mathbb{Z}^d$-action on a zero-dimensional compact abelian group X with dual module $M = \widehat{X}$.*

(1) *The following conditions are equivalent.*
  (a) *α is expansive;*
  (b) *the module M is Noetherian.*

(2) *The following conditions are equivalent.*
   (a) α_M *has positive entropy (with respect to the normalized Haar measure* λ_X *of* X*);*
   (b) $\alpha_{R_d/\mathfrak{p}}$ *has positive entropy for some* $\mathfrak{p} \in \mathrm{Asc}(M)$*;*
   (c) *some* $\mathfrak{p} \in \mathrm{Asc}(M)$ *is principal (and hence of the form* $\mathfrak{p} = (p) = pR_d$ *for some rational prime constant* $p > 1$*).*

(3) *For* $k \geq 2$*, the following conditions are equivalent.*
   (a) α_M *is mixing of order* k*;*
   (b) $\alpha_{R_d/\mathfrak{p}}$ *is mixing of order* k *for every* $\mathfrak{p} \in \mathrm{Asc}(M)$.

LEMMA 2.4: *Let* α *be an expansive zero-entropy algebraic* $\mathbb{Z}^d$*-action on a compact zero-dimensional abelian group* X*. Then there exists* $k \geq 2$ *such that for any closed non-trivial* α*-invariant subgroup* $Y \subset X$*, the restriction of* α *to* Y *is not mixing of order* k*.*

Proof: By the previous lemma $\mathrm{Asc}(\widehat{X})$ is finite and every $\mathfrak{p} \in \mathrm{Asc}(\widehat{X})$ is non-principal. For every $\mathfrak{p} \in \mathrm{Asc}(\widehat{X})$ we choose $f(\mathfrak{p}) \in R_d$ such that g.c.d. of the coefficients of $f(\mathfrak{p})$ is 1. Let $h \in R_d$ denote the product of these polynomials and let k denote the cardinality of the support of h. Suppose that Y is a closed non-trivial α-invariant subgroup of X and $\mathfrak{q}$ is an element of $\mathrm{Asc}(\widehat{Y})$. Since $\widehat{Y}$ is a quotient of $\widehat{X}$, it follows that $\mathfrak{q}$ contains some $\mathfrak{p} \in \mathrm{Asc}(\widehat{X})$, and in particular $h \in \mathfrak{q}$. Since the g.c.d. of the coefficients of h is 1, h does not belong to $p(\mathfrak{q})R_d$, which shows that the action $\alpha_{R_d/\mathfrak{q}}$ is not mixing of order k (cf. [3, Lemma 3.1]). Now the given assertion follows from the previous lemma. ∎

PROPOSITION 2.5: *Let* Γ *be a torsion-free discrete group which is not virtually abelian, and let* (X, α) *be a mixing algebraic* Γ*-action on a compact connected finite-dimensional abelian group* X*. Then for any* α*-invariant non-trivial closed subgroup* $H \subset X$*, the restriction of* α *to* H *is not mixing of order* $n+1$*, where* $n = \dim(X)$.

Proof: Without loss of generality we may assume that Γ is finitely generated. Since X is a compact connected finite-dimensional abelian group, we can identify $\widehat{X}$ with a subgroup of $\mathbb{Q}^n$ which contains $\mathbb{Z}^n$. We define $H^\perp \subset \widehat{X}$ by

$$H^\perp = \{\phi \in \widehat{X} \mid H \subset \ker(\phi)\}.$$

Then $\widehat{\alpha}$ can be viewed as a homomorphism from Γ to $\mathrm{Aut}(\widehat{X}) \subset \mathrm{GL}(n, \mathbb{Q})$ which leaves $H^\perp$ invariant. Since Γ is torsion-free, $\ker(\alpha) \subset \Gamma$ is either trivial or infinite. If $\ker(\alpha)$ is infinite, then it is easy to see that the action α is not mixing. Therefore, we may assume that Γ is a subgroup of $\mathrm{GL}(n, \mathbb{Q})$ and $\widehat{\alpha}$ is the natural action of Γ on $\widehat{X}$. By Tits' theorem Γ is either virtually solvable or contains a copy of F_2, the free group with two generators. In the former case we choose any finite-index solvable subgroup $\Gamma_0 \subset \Gamma$. Let G denote the Zariski closure of Γ_0 in $\mathrm{GL}(n, \mathbb{C})$ and let G_0 denote the connected component of G which contains 1. By Lie's theorem there exists $x \in \mathrm{GL}(n, \mathbb{C})$ such that xG_0x^{-1} is contained in the subgroup of upper triangular matrices. Hence the commutator subgroup $[G_0, G_0]$ has non-trivial fixed vectors in $\mathbb{C}^n$. We note that $\Gamma_1 = G_0 \cap \Gamma_0$ is a finite-index subgroup of Γ, and since Γ is not virtually abelian, $[\Gamma_1, \Gamma_1]$ is non-trivial. Let $W \subset \mathbb{C}^n$ denote the subspace consisting of fixed vectors of $[\Gamma_1, \Gamma_1]$. As W is nonzero and defined over $\mathbb{Q}$, there exists a nonzero element v in $W \cap \mathbb{Q}^n$. We note that for sufficiently large $N > 0$, $Nv \in \mathbb{Z}^n \subset \widehat{X}$ is fixed by $[\Gamma_1, \Gamma_1]$. Since this contradicts the assumption that α is mixing, we conclude that Γ is not virtually solvable. Therefore there exist A, B in $\Gamma \subset \mathrm{GL}(n, \mathbb{Q})$ which generate a free group. We define $C = ABA^{-1}B^{-1}$ and observe that C and B also generate a free group. Since $\det(C) = 1$, there exist integers $m_1, \ldots, m_n$ such that $\sum_{i=1}^{n} m_i C^i = I$. We define sequences $s_1, \ldots, s_n$ in Γ by $s_i(j) = B^{-j}C^iB^j$. Then for any non-zero ϕ in $\widehat{X}$ and for every $j \geq 1$,

$$\phi = \sum_{i=1}^{n} \alpha(s_i(j))(m_i\phi).$$

If β denotes the restriction of α to H then $\widehat{\beta}$ can be identified with the natural action of Γ on $\widehat{X}/H^\perp$. Since $H^\perp$ is a proper subgroup of $\widehat{X}$, by the above equation and Lemma 2.1, the action β is not mixing of order $n+1$. ∎

3. Rigidity of equivariant maps

For any compact abelian group X, we denote by $U(X, \mathbb{T})$ the group of all λ_X-equivalent classes of measurable maps from X to $\mathbb{T}$, together with pointwise multiplication. We denote by $\mathrm{End}(U(X, \mathbb{T}))$ the group of all endomorphisms of $U(X, \mathbb{T})$. For any $k > 0$, we denote by $X^{(k)}$ the product of k copies of X, and by $\partial_k \colon X^{(k)} \to \mathrm{End}(U(X, \mathbb{T}))$ we denote the map defined by

$$\partial_k(\mathbf{x})(f) = D_{x_1} \circ \cdots \circ D_{x_k}(f)$$

for every $\mathbf{x} = (x_1, \ldots, x_k)$ in $X^{(k)}$. Clearly, for any f in $U(X, \mathbb{T})$, $\partial_k(\mathbf{x})(f)$ is trivial if and only if f is a measurable polynomial from X to $\mathbb{T}$ and $\deg(f) \leq k-1$.

For any continuous automorphism θ of X, we denote the induced automorphism on $X^{(k)}$ by $\theta^{(k)}$. It is easy to see that $\partial_k(\mathbf{x})(f\circ\theta) = \partial_k(\theta^{(k)}\mathbf{x})(f)\circ\theta$.

We note the following elementary fact on measurable maps between compact abelian groups.

LEMMA 3.1: *Let X_1 and X_2 be compact abelian groups and let f be a measurable map from X_1 to X_2.*

1. *If there exists $k \geq 0$ such that for every ϕ in $\widehat{X_2}$ the map $\phi\circ f$ is a measurable polynomial of degree at most k, then f is a measurable polynomial and $\deg(f) \leq k$.*
2. *If for every ϕ in $\widehat{X_2}$ the map $\phi\circ f$ agrees λ_{X_1}-a.e. with a continuous map h_ϕ: $X_1 \to \mathbb{T}$, then there exists a continuous map h: $X_1 \to X_2$ such that $f = h$ a.e. λ_{X_1}.*

Proof: We fix $\mathbf{h}$ in $X^{(k+1)}$ and define B to be the set of all x in X_1 with the property that $\partial_{k+1}(\mathbf{h})(\phi\circ f)(x) \neq 1$ for some ϕ in $\widehat{X_2}$. Since $\widehat{X_2}$ is countable, from the given condition it follows that B has zero Haar measure. We note that for any ϕ in $\widehat{X_2}$ and for any x in $X \setminus B$,

$$\phi\circ\partial_{k+1}(\mathbf{h})(f)(x) = \partial_{k+1}(\mathbf{h})(\phi\circ f)(x) = 1.$$

Since characters separate points we conclude that $\partial_{k+1}(\mathbf{h})(f)(x) = 1$ on $X_1 \setminus B$, which proves the first assertion. To prove the second assertion we define $A \subset X_1$ by

$$A = \{x \in X_1 |\ \phi\circ f(x) \neq h_\phi(x) \text{ for some } \phi \text{ in } \widehat{X_2}\}.$$

From the given condition and from the countability of $\widehat{X_2}$ it follows that the set A has zero Haar measure. We note that for any χ, ψ in $\widehat{X_2}$ and for any x in $X_1 \setminus A$, $h_{\chi+\psi}(x) = h_\chi(x)\cdot h_\psi(x)$. Since each h_ϕ is continuous, we deduce that $h_{\chi+\psi} = h_\chi \cdot h_\psi$ for any χ, ψ in $\widehat{X_2}$. Hence for each x in X_1, the map $\phi \mapsto h_\phi(x)$ defines a homomorphism from $\widehat{X_2}$ to $\mathbb{T}$. By the duality theorem there exists map h: $X_1 \to X_2$ such that $h_\phi(x) = \phi\circ h(x)$ for all ϕ in $\widehat{X_2}$. It is easy to verify that the map h is continuous. We note that for any x in $X_1 \setminus A$ and for any ϕ in $\widehat{X_2}$, $\phi\circ f(x) = h_\phi(x) = \phi\circ h(x)$. Since characters separate points, this shows that $f(x) = h(x)$ for all x in $X_1 \setminus A$. ∎

We define a metric on $U(X,\mathbb{T})$ by $d(f_1, f_2) = ||f_1 - f_2||_2$, where $||.||_2$ denotes the L^2-norm on $U(X,\mathbb{T})$. It is easy to see that the multiplication in $U(X,\mathbb{T})$ is continuous with respect to this metric.

LEMMA 3.2: *Let X be a compact abelian group and let f be an element of $U(X,\mathbb{T})$. Then for any $k>0$, the map $\mathbf{x}\mapsto\partial_k(\mathbf{x})(f)$ from $X^{(k)}$ to $U(X,\mathbb{T})$ is continuous.*

Proof: First we will consider the special case when $k=2$. For any q in $U(X,\mathbb{T})$ let $\overline{q}$ denote the complex conjugate of q, and for any x in X let f_x denote the element of $U(X,\mathbb{T})$ defined by $f_x(y)=f(yx)$. We define maps $S_1,\ldots,S_4\colon X^{(2)}\to U(X,\mathbb{T})$ by

$$S_1(x_1,x_2)=f_{x_1x_2},\ S_2(x_1,x_2)=\overline{f_{x_1}},\ S_3(x_1,x_2)=\overline{f_{x_2}},\ S_4(x_1,x_2)=f.$$

We note that for all $\mathbf{x}\in X^{(2)}$, $\partial_2(\mathbf{x})(f)=S_1(\mathbf{x})\cdot S_2(\mathbf{x})\cdot S_3(\mathbf{x})\cdot S_4(\mathbf{x})$. Since the right regular representation of X on $L^2(X)$ is continuous, it follows that each S_i is a continuous map. Since multiplication is a continuous map in $U(X,\mathbb{T})$, this proves the given assertion in the case $k=2$. To prove the general case we define $S_1,\ldots,S_{2^k}$ in a similar way and apply the same argument. ∎

PROPOSITION 3.3: *Let X_1,X_2 be compact abelian groups and let $f\colon X_1\to X_2$ be a measurable polynomial. Then f agrees λ_{X_1}-a.e. with a continuous map. Furthermore, if X_1 is connected then f agrees λ_{X_1}-a.e. with an affine map.*

Proof: From Lemma 3.1 we see that it is enough to consider the case when X_2 is a closed subgroup of $\mathbb{T}$. We will prove both assertions by induction on $m=\deg(f)$.

For $i\geq 0$, let $P_i\subset U(X_1,\mathbb{T})$ denote the topological space consisting of all measurable polynomials $p\colon X_1\to X_2$ of degree at most i, together with the subspace topology. It is obvious that P_0 consists of constant maps. If $f\in P_1$, then $D_h(f)\in P_0$ for all h in X_1, i.e., $f(hx)=c_hf(x)$ λ_{X_1}-a.e., where c_h is a constant depending on h. This shows that f is an eigenfunction of the right regular representation of X_1 on $L^2(X_1)$, i.e., that f is an affine map.

Thus, the given assertions hold if $\deg(f)\leq 1$. Since characters form an orthonormal basis of $L^2(X_1)$ we also deduce that P_1 is homeomorphic with $P_0\times\widehat{X_1}$, where $\widehat{X_1}$ is equipped with the discrete topology.

Let π denote the projection map from P_1 to $\widehat{X_1}$. If $\deg(f)\geq 2$, we consider the map $q\colon X_1^{(m-1)}\to\widehat{X_1}$ defined by

$$q(\mathbf{x})=\pi\circ\partial_{m-1}(\mathbf{x})(f).$$

By the previous proposition q is a continuous map. Since $\widehat{X}$ is discrete, the image of q is finite. Hence there exists a finite-index subgroup $K_1 \subset X_1^{(m-1)}$ such that q factors through $X_1^{(m-1)}/K_1$. We choose a finite-index subgroup $K \subset X_1$ such that $K^{(m-1)} \subset K_1$. Then $\partial_{m-1}(\mathbf{x})(f)$ lies in P_0 for all $\mathbf{x}$ in $K^{(m-1)}$. This implies that the restriction of f to K is a measurable polynomial of degree at most $m-1$. Let $Ky_1, \ldots, Ky_l$ be the cosets of K and for $i = 1, \ldots, l$ let $f_i: X_1 \to X_2$ be the map defined by $f_i(x) = f(y_i x)$. Since $\partial_{m-1}(\mathbf{x})(f_i)(z) = \partial_{m-1}(\mathbf{x})(f)(y_i z)$ for each i, we conclude that restriction of each f_i to K is a measurable polynomial of degree at most $m-1$. By the induction hypothesis restriction of each f_i to K agrees λ_{X_1}-a.e. with a continuous map, i.e., f agrees λ_{X_1}-a.e. with a continuous map.

If X_1 is connected then q is trivial, i.e., $\deg(f) \leq m-1$. By the induction hypothesis f agrees λ_{X_1}-a.e. with an affine map. ∎

Proof of Theorem 1.1: Let $K \subset \widehat{X_2}$ denote the subgroup consisting of all ϕ such that $\phi \circ f$ is a measurable polynomial and $\deg(\phi \circ f) \leq k-1$. Since $\phi \circ \alpha_2(\gamma) \circ f = \phi \circ f \circ \alpha_1(\gamma)$ for all $\gamma \in \Gamma$, it follows that K is invariant under the action $\widehat{\alpha_2}$.

Suppose that K is a proper subgroup of $\widehat{X_2}$. We define a closed α_2-invariant subgroup $H \subset X_2$ by

$$H = \{x \mid \phi(x) = 1 \text{ for all } \phi \text{ in } K\}.$$

Let β denote the restriction of α_2 to H. By our assumption the action β is not mixing of order $k+1$. We note that $\widehat{\beta}$ can be identified with the Γ-action on $\widehat{X_2}/K$ induced by $\widehat{\alpha_2}$. By Lemma 2.1 there exist $\phi_0, \phi_1, \ldots, \phi_k$ in $\widehat{X_2}$, sequences $s_1, \ldots, s_k$ in Γ and a sequence $\chi_1, \chi_2, \ldots$ in K such that $\phi_0 \notin K$, $s_i(j) \to \infty$ as $j \to \infty$ for all $i \in \{1, \ldots, k\}$, and

$$\phi_0 = \chi_j + \sum_{i=1}^{k} \phi_i \circ \alpha_2(s_i(j))$$

for all $j \geq 0$. Since f is Γ-equivariant this implies that, for all $j > 0$,

$$\phi_0 \circ f = (\chi_j \circ f) \cdot \prod_{i=1}^{k} \phi_i \circ f \circ \alpha_1(s_i(j)).$$

For any h in $U(X_1, \mathbb{T})$ we define a map $h^{(k)}: X_1^{(k)} \to \mathbb{R}^+$ by

$$h^{(k)}(\mathbf{x}) = ||\partial_k(\mathbf{x})(h) - 1||_2.$$

By Lemma 3.2 the map $h^{(k)}$ is continuous for any h in $U(X_1, \mathbb{T})$. We note that $h_1^{(k)} h_2^{(k)} \leq h_1^{(k)} + h_2^{(k)}$ and $(h \circ \theta)^{(k)} = h^{(k)} \circ \theta^{(k)}$ for any continuous automorphism θ of X_1. Furthermore, $h^{(k)}$ is identically zero if and only if h is a measurable polynomial and $\deg(h) \leq k-1$. We define continuous maps $q_0, q_1, \ldots, q_k\colon X_1^{(k)} \to \mathbb{R}^+$ by $q_i = (\phi_i \circ f)^{(k)}$. Since for each j, the map $\chi_j \circ f$ is a measurable polynomial with degree at most $k-1$, it follows that $(\chi_j \circ f)^{(k)}$ is identically zero for all j. This implies that for all $j > 0$,

$$q_0 \leq \sum_{i=1}^{k} q_i \circ \alpha_1(s_i(j))^{(k)}.$$

We claim that q_0 is identically zero. It is enough to show that for any open set $U \subset X_1^{(k)}$ and for any $\epsilon > 0$ there exists $\mathbf{y} \in U$ such that $q_0(\mathbf{y}) \leq \epsilon$. We choose open sets $V_1, \ldots, V_k$ in X_1 such that $V_1 \times \cdots \times V_k \subset U$. It is easy to see that each q_i is uniformly continuous and has the property that $q_i(x_1, \ldots, x_k) = 0$ whenever $x_j = 1$ for some j in $\{1, \ldots, k\}$. Hence there exists a neighbourhood W of 1 in X_1 such that for each i, $q_i(\mathbf{x}) < \epsilon/2k$, whenever $x_j \in W$ for some j. As the action α is mixing, for any i the set $\alpha(s_i(j))(V_i) \cap W$ is non-empty for sufficiently large j. We choose $N > 0$ and $y_1, \ldots, y_k$ in X_1 such that for all i, $y_i \in V_i$ and $\alpha(s_i(N))(y_i) \in W$. Since $\mathbf{y} \in U$ and

$$q_0(\mathbf{y}) \leq \sum_{i=1}^{k} q_i \circ \alpha(s_i(N))^{(k)}(\mathbf{y}) \leq \epsilon/2,$$

this proves the claim. Now from the above claim it follows that $\phi_0 \circ f$ is a measurable polynomial with degree at most $k-1$, i.e., that $\phi_0 \in K$. This contradiction shows that K can not be a proper subgroup of $\widehat{X_2}$. Now Theorem 1.1 follows from Lemma 3.1. ∎

Proofs of Corollary 1.2 and Corollary 1.3: From Proposition 2.5 and Proposition 3.3 we see that Corollary 1.3 is an immediate consequence of Theorem 1.1. To prove Corollary 1.2, we fix any ϕ in $\widehat{X_2}$ and define K to be the smallest $\widehat{\alpha_2}$-invariant subgroup of $\widehat{X_2}$ which contains ϕ. Let β denote the $\mathbb{Z}^d$-action on $\widehat{K}$ induced by α_2, and let $\pi\colon X_2 \to \widehat{K}$ denote the dual of the inclusion map $i\colon K \to \widehat{X_2}$. Then π is a surjective $\mathbb{Z}^d$-equivariant continuous homomorphism from (X_2, α_2) to $(\widehat{K}, \beta)$. By Lemma 2.3 the action β is expansive. Since the action $(\widehat{K}, \beta)$ is a factor of (X_2, α_2), it has zero entropy. Applying Lemma 2.4 and Theorem 1.1 we see that the map $\pi \circ f$ is a measurable polynomial. Since ϕ is arbitrary, this implies that $\phi \circ f\colon X_1 \to \mathbb{T}$ is a measurable polynomial for all

ϕ in $\widehat{X_2}$. By Proposition 3.3 and Lemma 3.1 the map f agrees λ_{X_1}-a.e. with a continuous map. ∎

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